Penny Problem

Solution by Ed Cottrell

Problem: You have 12 pennies, 11 of which are of equal weight and one of which is counterfeit and weighs more or less than all the others. You are given a two-pan balance. Determine in three weighings which coin is different.

Solution:
I need to introduce some notation. The numbers {1,2,…12} represent the coins themselves. The letters {A,B,C} represent the weighings. X is the unusual coin. Order is important here (the first stack listed is the "left" stack, and the second is the "right" stack).

STAGE 1

Weigh the stacks 1,2,3,4 and 5,6,7,8. Call this weighing A.

Case I. They are even, so X is in {9,10,11,12} and {1,2,…,8} consists of good coins. Weigh the stacks 1,2,3 and 9,10,11 and call this weighing B.

Case i. They are even again, so X is **12**. Weigh 1 against 12 to determine whether 12 is unusually heavy or unusually light.

Case ii. They are not even, so X is in {9,10,11}. Weigh 9 against 10 and call this weighing C. If they are even, X is **11**. If not, then you know based on B whether X is unusually heavy or unusually light, enabling you to figure out which is X. To be precise, if B=C, X is **10**. If not, X is **9**.

Case II. They are different, so X is in {1,2,…,8} and {9,10,11,12} consists of good coins. Proceed to Stage 2.

STAGE 2

Weigh the stacks 1,2,5 and 3,4,6 and call this weighing B.

Case III. They are even, so X is in {7,8}. Weigh 1 against 7. If they are even, X is **8**. If not, X is **7**.

Case IV. They are uneven, and A=B (the balance tipped the same way). So, X didn't move this time, meaning X is in {1,2,6}. Switch things up by weighing 1 against 2 (this splits {1,2,6} into "left," "right," and "out" piles) and call this weighing C.

Case i. They are even. X is **6**.

Case ii. They are uneven, and A=B=C. Thus, X never moved once. The only possibility is X = **1**.

Case iii. They are uneven, and A=B=/=C. So, X did not move the first time but did the second. X = **2**.

Case V. They are uneven and A=/=B (the balance reversed). So, X moved, so X is in {3,4,5}. Weigh 3 against 4 (again, giving you three stacks).

Case i. They are even. X is **5**.

Case ii. They are not even, and A=/=B=C. So, X moved the first time and not the second. X must be **4**.

Case iii. They are not even, and A=/=B=/=C. So, X moved both times. X must be **3**.

There you have it. Breakdown by case:

X	CASE
1	IV. ii.
2	IV. iii.
3	V. iii.
4	V. ii.
5	V. i.
6	IV. i.

X	CASE
7	III (uneven)
8	III (even)
9	I. ii. (uneven)
10	I. ii. (even)
11	I. i. (uneven)
12	I. i. (even)

Note: By considering which way the balance actually tipped, it's possible to determine whether X is heavier or lighter in every case. In the case X=12, we used our third weighing exclusively for this purpose.